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Hopf defects as seeds for monopole loops

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Abstract: We investigate the relation between instantons and monopoles in the Laplacian Abelian Gauge using analytical methods in the continuum. Our starting point is the fact that the 't Hooft instanton with its high symmetry leads to a pointlike defect with Hopf invariant one. In order to generalise this result we partly break the symmetry by a local perturbation. We find that for generic configurations near the 't Hooft instanton the defects become loops. The analytical results show explicitly that these defects are magnetic monopoles with unit charge. In addition, the monopoles are twisted to account for the instanton number of the background.

Keywords: Solitons Monopoles and Instantons, Confinement, QCD.

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Contents

1. Introduction

Topological objects are prominent examples of non-perturbative effects in quantum field theories. For Yang-Mills theories these are instantons and magnetic monopoles,[1] respectively. While the first are intimately connected to the gauge invariant topological density and responsible for chiral symmetry breaking [1], the latter are visible only after an abelian gauge fixing [2], and supposed to be responsible for confinement. Since both physical effects take place below the same critical temperature [3], a relation between instantons and monopoles is highly desirable but still not fully known. The first result in this direction is due to Rossi [4]: a static 't Hooft-Polyakov monopole can be built out of an array of instantons placed along the time axis. A similar construction exists for the caloron [5].

To see how instantons are built from monopoles we take the point of view of abelian projections (for a detailed prescription see [6]). An abelian gauge is a partial gauge fixing leaving the maximal abelian subgroup untouched. We will restrict ourselves to the gauge group SU(2), where the maximal abelian subgroup is simply U(1) embedded in terms of diagonal matrices. The needed gauge transformation is best described by the diagonalisation of an 'auxiliary Higgs field' ϕ in the adjoint representation. Defects occur where this field vanishes, i.e. the diagonalisation becomes ambiguous. Since this means solving three equations, generic defects in four dimensions form lines. Moreover, the normalised Higgs field $n = \phi/|\phi|$ perpendicular to those lines generically is a hedgehog. It can be diagonalised only at the expense of introducing a singular gauge field, the Dirac monopole. By charge conservation defects form closed lines, i.e. loops.

[1] and center vortices

Topological arguments enforce the existence of defects for every configuration with non-vanishing instanton number on the four-sphere.[2] The topological properties of the defects necessary to generate an instanton number are highly non-trivial for general abelian gauges [8]. However, the relation between instantons and *static* monopoles in the Polyakov gauge is well understood [9]–[15].

Much less is known analytically about defects in the Laplacian Abelian Gauge (LAG) [16, 17, 18]. The Higgs field of the LAG is defined as the lowest eigenvector of the gauge covariant laplacian in the background of A, $-\mathrm{D}^2[A]\phi = E_0\phi$. The LAG has turned out to be 'very useful' on the lattice in the sense that it shares abelian dominance with the Maximal Abelian Gauge (MAG) [19] but does not suffer from a severe Gribov problem [20, 21]. Monopole loops have been observed for instanton backgrounds in the LAG by numerical means [22, 23]. A fully analytical treatment, however, is very difficult. So far it was only possible for the 't Hooft instanton [24] (and the meron [22]) which is highly symmetric and thus non-generic; the defect is degenerate to a point and localised at the instanton core (see below).

The present work is the first step towards an analytical investigation of generic configurations in the LAG. By breaking the high symmetry, we show that for configurations near the 't Hooft instanton the defect manifold becomes a loop (even a circle, section 2). Furthermore, the associated hedgehog is twisted once along the loop (the simplest possibility to account for the instanton number, section 3). As a by-product, the picture of Hopf defects as 'monopole loops with vanishing radius' is proven.

2. From Hopf defects to monopole loops

The ground state of the SU(2) covariant laplacian in the background of a 't Hooft instanton in regular gauge is of the form [24],

$$\phi = f(r) n_{\mathrm{H}}\,, \qquad f(r) \overset{r\to 0}{\longrightarrow} r^2\,, \tag{2.1}$$

where n_{H} is the standard Hopf map[3] [26, 27, 28]

$$n_{\mathrm{H}} \equiv \begin{pmatrix} 2(\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4) \\ 2(\hat{x}_2\hat{x}_3 - \hat{x}_1\hat{x}_4) \\ \hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3^2 - \hat{x}_4^2 \end{pmatrix}\,, \qquad \hat{x}_\mu \equiv x_\mu / r\,. \tag{2.2}$$

This Higgs field ϕ vanishes quadratically[4] at the instanton core — the origin — where a pointlike defect is located. We conjecture that this behaviour is not a feature of

[2]Such a strong statement does not hold for the four-torus as is plausible from the existence of *abelian instantons* [7].

[3]being the projection in the Hopf bundle describing the Dirac monopole [25]

[4]in agreement with lattice simulations [29]

the particular abelian gauge chosen, but rather a matter of symmetry. The 't Hooft instanton is spherically symmetric (in a proper definition involving gauge transformations); hence any abelian gauge which does not break the rotational symmetry SO(4) enforces the Higgs field to be spherically symmetric as well. Monopole loops would break this symmetry, while pointlike defects (as well as S^3 defect manifolds) do not.

On the other hand it is very easy to verify that the Higgs field (2.2) has the right topological behaviour. Living in an associated bundle it must have the same boundary conditions[5] as the gauge field,

$$r \to \infty : \qquad A \to ig\mathrm{d}g^\dagger \,, \quad n \to g\,\mathrm{const}\,g^\dagger \,. \tag{2.3}$$

For the instanton in regular gauge we have $g = h \equiv \hat{x}_4 \mathbb{1}_2 + i\hat{x}_a\sigma_a$. It follows that n, being a mapping from S^3 (in coordinate space) to S^2 (in color space), must have a Hopf invariant equal to the instanton number (equal to the winding of h). $n_\mathrm{H} = h\,\sigma_3/2\,h^\dagger$ is just the prototype mapping with Hopf invariant one.

Let us now *slightly perturb* the 't Hooft instanton, $A = A_\mathrm{inst} + \lambda\delta A$, with perturbation parameter λ. The usual Schrödinger perturbation theory for the change of the groundstate, $\phi = \phi_\mathrm{inst} + \lambda\delta\phi$, requires access to all eigenvalues and eigenfunctions of $-\mathrm{D}^2[A_\mathrm{inst}]$. These are not known analytically. But if perturbation theory is valid, the size of the defect manifold (i.e. the size of the expected monopole loop) is small. Therefore we can restrict ourselves to the vicinity of the origin. There we can Taylor expand $\delta\phi$; for our purposes even the lowest order approximation is sufficient. Thus the Higgs field of a generic configuration A *close to the instanton* (in orbit space) and *near the origin* (in coordinate space) is (cf. (2.1)),

$$\phi = \phi_\mathrm{inst} + \lambda\delta\phi = r^2\,n_\mathrm{H} + R^2\,\mathrm{const}\,, \tag{2.4}$$

where we have introduced a radius parameter R, since the Higgs field in our convention is of dimension $(\mathrm{length})^2$.

Without loss of generality we specialise to a perturbation pointing in the third color direction,

$$\phi = r^2\,n_\mathrm{H} - R^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{2.5}$$

All other cases can be obtained by rotations[6]. A straightforward calculation shows

[5] in the bundle language the same transition functions

[6] Rotations also account for the following fact: Since the unperturbed ground state is threefold degenerate [24], one in principle has to perform degenerate perturbation theory. Apart from the perturbation, it also selects the unperturbed state ϕ_inst. The high symmetry of the problem, however, allows to rotate this (unknown) state to the one considered above. The analogue in three dimensions is that any linear combination of spherical harmonics with $l = 1$ can be obtained from a particular one, say Y_{10}, upon rotating the argument, which is tantamount to choosing another coordinate basis.

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that the zeros of ϕ are then on the *circle* $C: \ x_1^2 + x_2^2 = R^2$, $x_3 = x_4 = 0$. Its size scales with the perturbation parameter $R = \sqrt{\lambda}$ (see (2.4)). The perturbation has *enlarged the defect manifold from a point to a loop*, thereby breaking the spherical symmetry. Such a picture was conjectured in [30] for instantons in the MAG, but there the formation of the loop is suppressed by the gauge fixing functional [31].

Notice that the deformed Higgs field ϕ is now on a different orbit, since we changed its zeros which are gauge invariant. Its global properties, however, remain the same (as we will also see in the next section) because we performed only a local perturbation around the origin.

3. Monopole charge and twist

To proceed further we introduce polar angles for both coordinate space $\mathbb{R}^4$,

$$x = (r_{12}\cos\varphi_{12}\,, r_{12}\sin\varphi_{12}\,, r_{34}\cos\varphi_{34}\,, r_{34}\sin\varphi_{34})\,,\ r_{12} = r\cos\vartheta,\ r_{34} = r\sin\vartheta\,, \tag{3.1}$$

and color space $\mathbb{R}^3$ resp. S^2,

$$n = \begin{pmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{pmatrix}. \tag{3.2}$$

The Hopf map (2.2) is given by assigning

$$\alpha_{\rm H} = \varphi_{12} - \varphi_{34}\,, \qquad \beta_{\rm H} = 2\vartheta = \arctan\frac{2r_{12}r_{34}}{r_{12}^2 - r_{34}^2}\,, \tag{3.3}$$

while the perturbation (2.5) corresponds to a deformation of β,

$$\alpha = \varphi_{12} - \varphi_{34}\,, \qquad \beta = \arctan\frac{r^2\sin(2\vartheta)}{r^2\cos(2\vartheta) - R^2} = \arctan\frac{2r_{12}r_{34}}{r_{12}^2 - r_{34}^2 - R^2}\,. \tag{3.4}$$

It turns out that this Higgs field perfectly agrees with the one considered in [30, 8], $\beta = \vartheta_+ + \vartheta_-$, $\tan\vartheta_\pm = r_{34}/(r_{12} \pm R)$.

From figure 1 it is obvious that both Higgs fields (3.3) and (3.4) agree in their global ($r \to \infty$) properties. For the local properties of the new field it is important to notice that on the loop C both angles α and β are singular. In a vicinity perpendicular to the loop they take on all values $\alpha \in [0, 2\pi]$, $\beta \in [0, \pi]$. Put differently, the normalised field n is homotopic to the hegdehog: on any two-sphere perpendicular to the loop it covers the whole two-sphere in color space exactly once. Viewed as a mapping $n:\ S^2 \to S^2$ it has winding number one. From the 't Hooft-Polyakov monopole it is well-known that such a Higgs field cannot be diagonalised[7] smoothly [32].

[7] or brought to 'unitary gauge'

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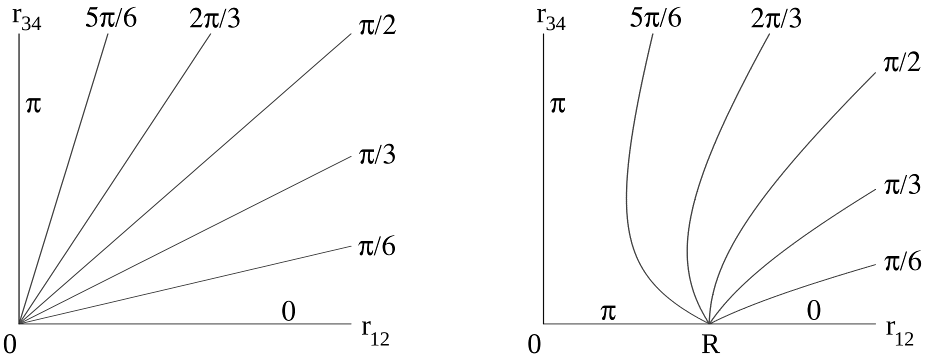

Figure 1: Lines of constant β as a function of the two radial coordinates r_{12} and r_{34} for the Hopf defect (left, cf. (3.3)) and a monopole loop at $r_{12} = R$, $r_{34} = 0$ (right, essentially copied from [30, 8], cf. (3.4)). In both cases the remaining polar angle is $\alpha = \varphi_{12} - \varphi_{34}$.

One may nevertheless diagonalise n by a singular gauge transformation; in this way the Dirac monopole appears in the gauge field. The relevant gauge transformation is

$$g = e^{i\gamma\sigma_3/2} e^{i\beta\sigma_2/2} e^{i\alpha\sigma_3/2} . \tag{3.5}$$

The residual U(1)-freedom (of rotations around the third color axis) is encoded in γ. We choose

$$\gamma = \varphi_{12} + \varphi_{34} , \tag{3.6}$$

for which g is singular on the disc $D: \ r_{12} \leq R,\ r_{34} = 0$, spanned by the loop C.

The gauge transformation (3.5) induces an inhomogeneous term, the abelian part of which is

$$a \equiv (i\Omega \mathrm{d}\Omega^{-1})_3 = \mathrm{d}\gamma + \cos\beta \mathrm{d}\alpha . \tag{3.7}$$

One can easily compute the abelian field strength,

$$f \equiv \mathrm{d}a = f^{\mathrm{reg}} + f^{\mathrm{sing}} , \tag{3.8}$$

$$f^{\mathrm{reg}} = -\sin\beta\, \mathrm{d}\beta \wedge \mathrm{d}\alpha , \tag{3.9}$$

$$f^{\mathrm{sing}} = (1 - \cos\beta)\, \mathrm{d}^2\varphi_{34} = 4\pi\theta(R - r_{12})\delta(r_{34})\mathrm{d}x_3 \wedge \mathrm{d}x_4 . \tag{3.10}$$

The regular part f^{reg} is just the Coulombic magnetic field, while the singular part f^{sing} is the set of all Dirac strings filling the disc D, called the Dirac sheet [33]. We use the latter to identify the monopoles as endpoints of Dirac strings. The magnetic current is,

$$k \equiv *\mathrm{d}f^{\mathrm{sing}} = 4\pi\delta(r_{12} - R)\delta(r_{34})\mathrm{d}\varphi_{12} . \tag{3.11}$$

The angle α not only depends on φ_{34} (which gives the hedgehog) but also on the world-line coordinate φ_{12}. This means that the monopole is 'twisted' [34] once: the

Higgs field n rotates once around the third axis in isospace while moving along the loop. Our Higgs field is such that the complicated relation in [8] reduces to

$$\text{Hopf invariant} = \text{magnetic charge} \times \text{twist}\,, \qquad 1 = 1 \times 1\,. \tag{3.12}$$

Gradually 'switching off' the perturbation $\lambda\delta\phi$ one can see that *a Hopf defect emerges when a twisted monopole loop is shrunk to vanishing radius.* The Dirac sheet D degenerates to a point, too. Notice that the two-spheres used to measure the magnetic charge are incapable to detect the Hopf defect. Instead one has to pass to three-spheres surrounding a point in four dimensions.

4. Conclusions

We have investigated the Laplacian Abelian Gauge in the vicinity of the 't Hooft instanton by means of Schrödinger perturbation theory. While the spherically symmetric instanton is related to a pointlike defect, a generic (constant) perturbation induces a monopole loop with unit charge and twist (cf. (3.4) and its interpretation). Together these topological quantities give rise to the Hopf invariant, which reflects the instanton number of the background gauge field. Extending the correlation between the defect manifold and the instanton core in the unperturbed case, the instanton density of the new background is supposed to be localised on a circle [35, 36]. Our result implies that for isolated, highly symmetric instantons the monopole loops are very small. Such small loops could be missed in lattice simulations of the abelian and monopole string tension.

Since isolated instantons are not sufficient for confinement, the defects induced by them cannot be the whole story. Bringing more instantons and anti-instantons close to each other, monopole loops start to spread out [30, 22]. Percolation is achieved if there are monopole loops extending over the whole space-time as verified in lattice simulations [37, 38]. Further analytic approaches are necessary to better understand this mechanism.

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References

[1] T. Schäfer and E.V. Shuryak, *Instantons in QCD*, *Rev. Mod. Phys.* **70** (1998) 323 [hep-ph/9610451].

[2] G. 't Hooft, *Topology of the gauge condition and new confinement phases in nonabelian gauge theories*, *Nucl. Phys.* **B 190** (1981) 455.

[3] J.B. Kogut et al., *Deconfinement and chiral symmetry restoration at finite temperatures in* SU(2) *and* SU(3) *gauge theories*, *Phys. Rev. Lett.* **50** (1983) 393.

[4] P. Rossi, *Propagation functions in the field of a monopole*, *Nucl. Phys.* **B 149** (1979) 170.

[5] T.C. Kraan and P. van Baal, *Periodic instantons with non-trivial holonomy*, *Nucl. Phys.* **B 533** (1998) 627 [hep-th/9805168].

[6] G. 't Hooft, *Monopoles, instantons and confinement*, hep-th/0010225.

[7] G. 't Hooft, *Some twisted selfdual solutions for the Yang-Mills equations on a hypertorus*, *Comm. Math. Phys.* **81** (1981) 267.

[8] O. Jahn, *Instantons and monopoles in general abelian gauges*, *J. Phys.* **A33** (2000) 2997 [hep-th/9909004].

[9] N. Weiss, *The effective potential for the order parameter of gauge theories at finite temperature*, *Phys. Rev.* **D 24** (1981) 475.

[10] H.W. Griesshammer, *Magnetic defects signal failure of abelian projection gauges in QCD*, hep-ph/9709462.

[11] H. Reinhardt, *Resolution of Gauss' law in Yang-Mills theory by gauge- invariant projection: topology and magnetic monopoles*, *Nucl. Phys.* **B 503** (1997) 505 [hep-th/9702049].

[12] C. Ford, U.G. Mitreuter, T. Tok, A. Wipf and J.M. Pawlowski, *Monopoles, Polyakov loops and gauge fixing on the torus*, *Annals Phys.* **269** (1998) 26 [hep-th/9802191].

[13] O. Jahn and F. Lenz, *Structure and dynamics of monopoles in axial gauge QCD*, *Phys. Rev.* **D 58** (1998) 085006 [hep-th/9803177].

[14] C. Ford, T. Tok and A. Wipf, *Abelian projection on the torus for general gauge groups*, *Nucl. Phys.* **B 548** (1999) 585 [hep-th/9809209].

[15] C. Ford, T. Tok and A. Wipf, SU(N)*-gauge theories in Polyakov gauge on the torus*, *Phys. Lett.* **B 456** (1999) 155 [hep-th/9811248].

[16] A.J. van der Sijs, *Laplacian abelian projection*, *Nucl. Phys.* **53** *(Proc. Suppl.)* (1997) 535 [hep-lat/9608041].

[17] A.J. van der Sijs, *Abelian projection without ambiguities*, *Prog. Theor. Phys. Suppl.* **131** (1998) 149 [hep-lat/9803001].

[18] A.J. van der Sijs, *Laplacian abelian projection: abelian dominance and monopole dominance*, *Nucl. Phys.* **73** *(Proc. Suppl.)* (1999) 548 [hep-lat/9809126].

[19] E.M. Ilgenfritz, S. Thurner, H. Markum and M. Muller-Preussker, *Monopole characteristics in various abelian gauges*, *Phys. Rev.* **D 61** (2000) 054501 [hep-lat/9904010].

[20] G.S. Bali, V. Bornyakov, M. Müller-Preußker and K. Schilling, *Dual superconductor scenario of confinement: a systematic study of Gribov copy effects*, *Phys. Rev.* **D 54** (1996) 2863 [hep-lat/9603012].

[21] P. de Forcrand and M. Pepe, *Center vortices and monopoles without lattice Gribov copies*, *Nucl. Phys.* **B 598** (2001) 557 [hep-lat/0008016].

[22] H. Reinhardt and T. Tok, *Abelian and center gauges in continuum Yang-Mills theory*, hep-th/0009205.

[23] P. de Forcrand and M. Pepe, *Laplacian gauge and instantons*, *Nucl. Phys.* **94** *(Proc. Suppl.)* (2001) 498 [hep-lat/0010093].

[24] F. Bruckmann, T. Heinzl, T. Vekua and A. Wipf, *Magnetic monopoles vs. Hopf defects in the laplacian (abelian) gauge*, *Nucl. Phys.* **B 593** (2001) 545 [hep-th/0007119].

[25] L. H. Ryder, *Dirac monopoles and the Hopf map* $S^3 \to S^2$, *J. Phys.* **A13** (1980) 437.

[26] H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, *Math. Ann.* **104** (1931) 637.

[27] M. Nakahara, *Geometry, topology and physics*, Adam Hilger, 1990.

[28] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern geometry - methods and applications,* Vol. 2, Springer, 1985.

[29] P. de Forcrand, private communication.

[30] R.C. Brower, K.N. Orginos and C.-I. Tan, *Magnetic monopole loop for the Yang-Mills instanton*, *Phys. Rev.* **D 55** (1997) 6313 [hep-th/9610101].

[31] F. Bruckmann, T. Heinzl, A. Wipf and T. Tok, *Instantons and Gribov copies in the maximally abelian gauge*, *Nucl. Phys.* **B 584** (2000) 589 [hep-th/0001175].

[32] J. Arafune, P.G. O. Freund and C.J. Goebel, *Topology of Higgs fields*, *J. Math. Phys.* **16** (1975) 433.

[33] P. A. M. Dirac, *The theory of magnetic poles*, *Phys. Rev.* **74** (1948) 817.

[34] C. H. Taubes, *Morse theory and monopoles: Topology in long-ranged forces*, in *Progress in gauge field theory*, G. 't Hooft, ed., Plenum Press, New York 1984.

[35] M. Garcia Perez, T.G. Kovacs and P. van Baal, *Overlapping instantons*, hep-ph/0006155.

[36] D. Hansen et. al., work in progress.

[37] V.G. Bornyakov, V.K. Mitrjushkin and M. Muller-Preussker, *Deconfinement transition and abelian monopoles in* SU(2) *lattice gauge theory*, *Phys. Lett.* **B 284** (1992) 99.

[38] T.L. Ivanenko, A.V. Pochinsky and M.I. Polikarpov, *Condensate of abelian monopoles and confinement in lattice gauge theories*, *Phys. Lett.* **B 302** (1993) 458.